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正定値行列の幾何構造について
On geometric structure of positive definite matrices

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In this note, from the viewpoint of Corach-Porta-Recht [3, 4], we discuss a Riemannian geometry for the $n\times n$ positive definite matrices $\mathcal{C}(n)$ by Bhatia-Holbrook [2], say the *CPRBH geometry*: The principal fiber bundle is the regular matrices $\mathcal{G}=\mathcal{G}(n)$ with the unitary group $\mathcal{U}(n)$ as the structure one and the projection $\pi(X)=XX^{*}$. The fiber at $A\in \mathcal{C}(n)$ is $\pi^{-1}(A)=\sqrt{A}\mathcal{U}(n)$ and the Rimannian metric $g_{A}(X, Y)=\mathrm{tr}\,(A^{-1}XA^{-1}Y)$ at $A\in \mathcal{C}(n)$. It was shown in [4] that the path of the geometric means

$$A\#_{t}B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{t}A^{\frac{1}{2}}$$

is the geodesic from A at $t=0$ to B at $t=1$.

The manifold $\mathcal{C}(n)$ is a homogeneous space $\mathcal{G}(n)/\mathcal{U}(n)$ with the involution $\sigma(T)=(T^{*})^{-1}$ for $T\in \mathcal{G}(n)$. The differential $d\sigma(Z)=-Z^{*}$ for $Z\in \mathcal{T}(\mathcal{G}(n))=\mathcal{M}_{n}$ is the Cartan involution with the Cartan decomposition as a Lie group and a Lie algebra;

$$\mathcal{G}(n)=\mathcal{U}(n)\mathcal{C}(n), \quad \mathfrak{gl}(n)=\mathfrak{u}(n)\oplus \mathcal{T}\mathcal{C}(n)=\mathfrak{u}(n)\oplus i\mathfrak{u}(n)$$

where the Lie algebra $\mathfrak{u}(n)$ is the skew-hermitian matrices and the tangent bundle $\mathcal{T}\mathcal{C}(n)$ is the hermitian ones. In fact, $d\sigma$ is the Cartan involution since

$$-B(X, d\sigma(X))=\mathrm{trad}X\mathrm{ad}X^{*}=2n\mathrm{tr}XX^{*}-2\mathrm{tr}X\mathrm{tr}X^{*}\geqq 0$$

where B is the Killing form.

It is related to the connection in $\mathcal{G}$: The vertical space in the tangent space $\mathcal{T}\pi^{-1}(A)$ is $\sqrt{A}U\mathfrak{u}(n)$ and the horizontal one is $\sqrt{A}U\mathcal{T}\mathcal{C}(n)$ for some unitary U. In fact, for an invertible matrix G, the orthogonal decomposition at $T=\sqrt{A}U$ is

$$G=\frac{T(T^{-1}G-G^{*}(T^{*})^{-1})}{2}+\frac{T(T^{-1}G+G^{*}(T^{*})^{-1})}{2}$$

Thereby the horizontal lift Γ of γ should satisfy that $\Gamma^{-1}\dot{\Gamma}$ is hermitian, i.e., the horizontal condition is

$$\dot{\Gamma}\Gamma^{*}=\Gamma\dot{\Gamma}^{*}.$$

Moreover, as Pálfia [12] pointed, $\mathcal{C}(n)$ is a symmetric space with the symmetry s_A at $A \in \mathcal{C}(n)$ satysfying $s_A(B) = AB^{-1}A$. The Cartan decomposition shows that a symmetric space $\mathcal{U}(n) = \mathcal{U}(n) \times \mathcal{U}(n)/\Delta\mathcal{U}(n)$ is the real form and its dual symmetric space $\mathcal{U}(n)_{\mathbb{C}}/\mathcal{U}(n)$ is $\mathcal{C}(n)$ itself where $\Delta\mathcal{U}(n)$ is the diagonal subspace and $\mathcal{U}(n)_{\mathbb{C}}$ is the complexification of $\mathcal{U}(n)$. This shows that it is not compact and the sectional curvature is non-positive, that is $\mathcal{C}(n)$ is a *CAT(0)-space*. Let γ and δ be geodesics. If

$$d(\gamma(1/2), \delta(1/2)) \leq \frac{d(\gamma(1), \delta(1))}{2}$$

always holds, then it is said that *Busemann curvatures are non-positive.* If

$$d^2(Z, \gamma(t)) \leqq (1-t)\left(d^2(Z, \gamma(0)) + td^2(Z, \gamma(1))\right) - t(1-t)d^2(\gamma(0), \gamma(1))$$

always holds, it is said that *Alexandrov curvatures are non-positive.* This inequality is called *Courbure négative one* or *semi-parallelogram law* for the case $t = 1/2$ ([1]). In the Riemannian case, they are equivalent to nonpositivity of sectional curvature [10]. Moreover, $\mathcal{C}(n)$ is a (simply connected) complete space, it is called *Hadamard manifold.* Then it is known that $F(t) = d(\gamma(t), \delta(t))$ is convex.

Since every symmetric space is geodesically complete (hence we also have that it is complete as a metric space), the extended curve

$$\gamma(t) = A\natural_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}$$

for $t \in (-\infty, \infty)$ is the geodesic including $A\#_t B$. Then we have the parallel translate along the geodesic is given by

Theorem. *One of the horizontal lift of the geodesic $\gamma(t) = A\natural_t B$ is*

$$\Gamma(t) = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{t}{2}}$$

and the parallel translate P_t^s from $\gamma(s)$ to $\gamma(t)$ along γ in the tangent bundle $\mathcal{TC}(n)$ is given by

$$P_t^s X = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{t-s}{2}} A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{t-s}{2}} A^{\frac{1}{2}}.$$

Proof. By

$$\pi(\Gamma(t)) = \Gamma(t)\Gamma(t)^* = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}} = \gamma(t),$$

Γ is a lift of γ. The horizontality follows from the fact that

$$2\Gamma(t)^{-1}\dot{\Gamma}(t) = 2\dot{\Gamma}(t)^*\Gamma^*(t)^{-1} = \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)$$

is hermitian. The parallel translate of X from s to t is

$$\begin{aligned} P_t^s X &= \Gamma(t)\Gamma(s)^{-1}X(\Gamma(s)^{-1})^*\Gamma(t)^* \\ &= A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{t-s}{2}} A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{t-s}{2}} A^{\frac{1}{2}}. \qquad \square \end{aligned}$$

Condider the triangle closed path $I \xrightarrow{A^t} A \xrightarrow{A\#_t B} B \xrightarrow{B^{1-t}} I$. Then the parallel translate of X isV^*XV for

$$V = A^{\frac{1}{2}}A^{-\frac{1}{2}}C^{\frac{1}{2}}A^{\frac{1}{2}}B^{-\frac{1}{2}} = A^{\frac{1}{2}}A^{-\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}B^{-\frac{1}{2}} = \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}B^{-\frac{1}{2}}.$$

Thus, $V^*V = I$ and $\det V = (\det A)^0(\det B)^0 = 1$, so that $V \in \mathcal{SU}(n)$. Approximating any loop by a polygon of geodesics, we have:

Corollary. *The holonomy group of $\mathcal{C}(n)$ is included by $\mathcal{SU}(n)$.*

Remark. In virtue of the Ambrose-Singer theorem, Pálfia [12] showed that they coinside via the Lie algebra $\mathfrak{su}(n)$, which might be already known.

In this geometry, the tangent vector at $\gamma(t)$ is given by (cf. [9])

$$S_t(A|B) = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

in particular, the tangent one at $t = 0$ is the relative operator entropy [5, 6]:

$$S(A|B) = S_0(A|B) = A^{\frac{1}{2}}\log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

For the above lift Γ, the horizontal condition is now

$$2\Gamma(t)^{-1}\dot{\Gamma}(t) = 2\dot{\Gamma}(t)^*\Gamma^*(t)^{-1} = A^{-\frac{1}{2}}S(A|B)A^{-\frac{1}{2}}.$$

Recently E.Kamei pointed in a seminar talk that the tangent vector at r

$$S_r(A|B) = (A\natural_r B)(A\natural_t B)^{-1}S_t(A|B).$$

shows the parallel translate of the tangent vector $S_t(A|B)$ to $S_r(A|B)$. In fact, for $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\begin{aligned} \Gamma(r)\Gamma(t)^{-1}S_t(A|B)\Gamma(t)^{-1}\Gamma(r) &= A^{\frac{1}{2}}C^{\frac{r-t}{2}}A^{-\frac{1}{2}}S_t(A|B)A^{-\frac{1}{2}}C^{\frac{r-t}{2}}A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}C^{\frac{r-t}{2}}C^t(\log C)C^{\frac{r-t}{2}}A^{\frac{1}{2}} = A^{\frac{1}{2}}C^r\log C\,A^{\frac{1}{2}} = S_r(A|B). \end{aligned}$$

In Hadamard manifolds, the *parallel* geodesics are defined by the boundedness;

$$d(\gamma(t), \delta(t)) < {}^{\exists}M$$

for all $t \in \mathbb{R}$ (it is also called *asymptotic*). But the parallel translates for the parallel vectors along parallel geodesics are not always parallel. So, considering flat geometry in $\mathcal{C}(n)$, we need Γ-commutativity ([2]): A, B and C are Γ-*commute* if matrices $C^{-\frac{1}{2}}AC^{-\frac{1}{2}}$, $C^{-\frac{1}{2}}BC^{-\frac{1}{2}}$ commute. It is equivalent to the commutativity of matrices

$$X^{-\frac{1}{2}}AX^{-\frac{1}{2}},\ X^{-\frac{1}{2}}BX^{-\frac{1}{2}},\ X^{-\frac{1}{2}}CX^{-\frac{1}{2}}$$

for some X.

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